ACADEMIC ASSOCIATES
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MATHEMATICS SPECIALIST

Calculator-assumed

UNIT 1 Semester 1 2023

Marking Key

Marking keys outline the expectations of examination responses. They help to ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section Two: Calculator Assumed (92 marks)

Question 8 (9 marks)

Given $x \in \mathbb{R}$, use the following statement to answer the questions below:

If $x^2 + 5x < 0$ then $x < 0$

(a) (i) Write down the converse of this statement. (1 mark)

Solution	Specific behaviours	Point
If $x < 0$ then $x^2 + 5x < 0$	✓ Writes down the converse.	1.1.1

(ii) Use a counter-example to prove the converse is false. (2 marks)

Solution	Specific behaviours	Point
If $x = -6$ then $x^2 + 5x = 36 - 30 = 6 \nless 0$	✓ Determines a suitable value of x. ✓ Shows the statement is false.	1.1.5

(b) (i) Write down the contrapositive of this statement. (1 mark)

Solution	Specific behaviours	Point
If $x \geq 0$ then $x^2 + 5x \geq 0$	✓ Writes down the contrapositive.	1.1.1

(ii) Prove the contrapositive is true. (2 marks)

Solution	Specific behaviours	Point
If $x \geq 0$ then $x^2 \geq 0$ and $5x \geq 0$ Hence $x^2 + 5x \geq 0$ $\therefore$ contrapositive is true	✓ Indicates individual terms of $x^2 + 5x$ are positive. ✓ Explains why statement is true.	1.1.1

(c) Hence, determine the truth of the original statement. Justify your answer. (2 marks)

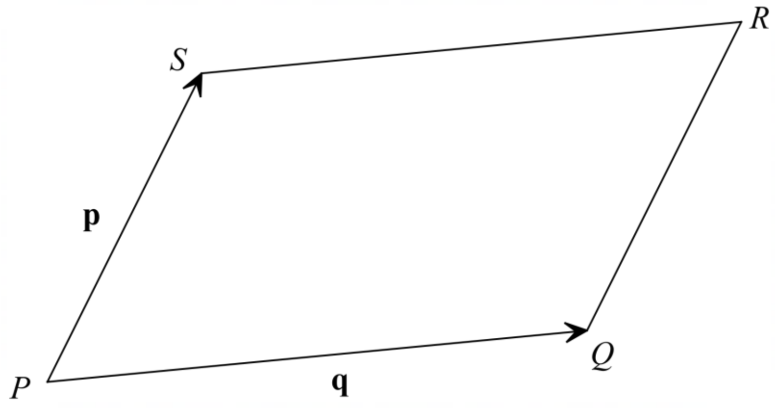
Solution	Specific behaviours	Point
As the contrapositive is true, then the original statement is true.	✓ States original statement is true. ✓ Explains using the contrapositive.	1.1.1

(d) Explain why the statement $x^2 + 5x < 0 \Leftrightarrow x < 0$ is false. (1 mark)

Solution	Specific behaviours	Point
To be true both the original statement and converse must be true, but the converse is fale.	✓ Explains using converse.	1.1.3

Question 9 (6 marks)

The diagram below shows parallelogram $PQRS$, with $\overrightarrow{PQ} = \mathbf{q}$ and $\overrightarrow{PS} = \mathbf{p}$.



(a) Write down a vector that represents $\overrightarrow{PR}$ into terms of $\mathbf{p}$ and $\mathbf{q}$. (1 mark)

Solution	Specific behaviours	Point
$\overrightarrow{PR} = \mathbf{p} + \mathbf{q}$	✓ Determines expression.	1.3.4

(b) Write down a vector that represents $\overrightarrow{SQ}$ into terms of $\mathbf{p}$ and $\mathbf{q}$. (1 mark)

Solution	Specific behaviours	Point
$\overrightarrow{SQ} = \mathbf{q} - \mathbf{p}$	✓ Determines expression.	1.3.4

(c) Determine an expression for $\overrightarrow{PR} \bullet \overrightarrow{SQ}$. (2 marks)

Solution	Specific behaviours	Point
$\overrightarrow{PR} \bullet \overrightarrow{SQ} = (\mathbf{p} + \mathbf{q}) \bullet (\mathbf{q} - \mathbf{p})$ $= \mathbf{p} \bullet \mathbf{q} - \lvert\mathbf{p}\rvert^2 + \lvert\mathbf{q}\rvert^2 - \mathbf{q} \bullet \mathbf{p}$ $= \lvert\mathbf{q}\rvert^2 - \lvert\mathbf{p}\rvert^2$	✓ Correctly expands out brackets. ✓ Simplifies expression.	1.3.10 1.3.12

(d) Describe geometrically shape $PQRS$, when $\overrightarrow{PR} \bullet \overrightarrow{SQ} = 0$, justifying your answer. (2 marks)

Solution	Specific behaviours	Point
$\lvert\mathbf{q}\rvert^2 - \lvert\mathbf{p}\rvert^2 = 0$ $\Rightarrow \lvert\mathbf{p}\rvert = \lvert\mathbf{q}\rvert$ $PQRS$ is a rhombus	✓ Recognises $\lvert\mathbf{p}\rvert = \lvert\mathbf{q}\rvert$. ✓ States $PQRS$ is a rhombus.	1.1.16

Question 10 (11 marks)

(a) A bag contains six counters numbered 4, 5, 6, 7, 8 and 9. The counters are picked out, one at a time, and placed in a row from left to right.

(i) How many six-digit numbers will have all the odd and all the even numbers together? (2 marks)

Solution	Specific behaviours	Point
$3! \times 3! \times 2 = 72$	✓ Arrangements of odd and even numbers. ✓ Total arrangements.	1.2.1

(ii) How many six-digit numbers will contain the digits 789 in that order? (2 marks)

Solution	Specific behaviours	Point
$4! = 24$	✓ Determines answer.	1.2.1

(iii) How many six-digit numbers will **not** have the prime numbers adjacent? (2 marks)

Solution	Specific behaviours	Point
$5! \times 2!$ $= 240$ have 5 and 7 adjacent $6! - 240$ $= 480$ don't have 5, 7 adjacent	✓ Determines arrangements with 5 and 7 adjacent. ✓ Determines answer.	1.1.2

(iv) If only three counters were picked out, one at a time, and placed in a row from left to right, how many of these numbers will have the digits in ascending order? (1 mark)

Solution	Specific behaviours	Point
$\binom{6}{3} = 20$	✓ Determines number.	1.2.7

(b) At a mathematics competition, there are 4 students from Alpha School, 7 from Beta College and 5 from Gamma School. Only 6 students can present their solutions, with each school picking 2 students.

How many different orders are possible,

(i) if students are selected at random to present? (2 marks)

Solution	Specific behaviours	Point
$\binom{4}{2}\binom{7}{2}\binom{5}{2} \times 6! = 907200$	✓ Chooses two students from each school. ✓ Determines arrangements.	1.2.1 1.2.7

(ii) if students from the same school must present one after the other? (2 marks)

Solution	Specific behaviours	Point
$\binom{4}{2} \times 2! \times \binom{7}{2} \times 2! \times \binom{5}{2} \times 2! \times 3!$ $= 60480$	✓ Chooses two students from each school and arrangements them. ✓ Determines total arrangements.	1.2.1 1.2.7

Question 11 (13 marks)

Vectors a and b are defined as $\mathbf{a} = -3\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} - 6\mathbf{j}$.

(a) Determine $3\mathbf{a} - \mathbf{b}$. (1 mark)

Solution	Specific behaviours	Point
$3\mathbf{a} - \mathbf{b} = -15\mathbf{i} + 9\mathbf{j}$	✓ Determines expression.	1.3.8

(b) Determine the value of scalars x and y if $x\mathbf{a} + y\mathbf{b} = -36\mathbf{i} + 32\mathbf{j}$. (2 marks)

Solution	Specific behaviours	Point
$\mathbf{i}$ components: $-3x + 6y = -36$ $\mathbf{j}$ components: $x - 6y = 32$ $x = 2, y = -5$	✓ Compares $\mathbf{i}$ and $\mathbf{j}$ components. ✓ Determines x and y.	1.3.8 1.3.9

(c) Find a vector with the same length of $\mathbf{a}$ in the direction of $\mathbf{b}$. (3 marks)

Solution	Specific behaviours	Point
$\lvert\mathbf{a}\rvert = \sqrt{10}$ $\hat{\mathbf{b}} = \frac{1}{6\sqrt{2}}(6\mathbf{i} - 6\mathbf{j}) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ Required vector is $\sqrt{5}(\mathbf{i} - \mathbf{j})$	✓ Determines magnitude of $\mathbf{a}$. ✓ Determines unit vector $\hat{\mathbf{b}}$. ✓ Determines required vector.	1.3.7

(d) Determine the vector projection of $\mathbf{a}$ on $\mathbf{b}$. (2 marks)

Solution	Specific behaviours	Point
$(\mathbf{a} \bullet \hat{\mathbf{b}})\hat{\mathbf{b}} = -2\sqrt{2}\left(\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})\right) = -2\mathbf{i} + 2\mathbf{j}$	✓ Determines dot product of $\mathbf{a} \bullet \hat{\mathbf{b}}$. ✓ Determines vector projection.	1.3.13

(e) The vectors $\mathbf{b}$ and $\mathbf{c} = 3\mathbf{i} + x\mathbf{j}$ are of equal length. Determine x. (2 marks)

Solution	Specific behaviours	Point
$6\sqrt{2} = \sqrt{9 + x^2}$ $x = \pm 3\sqrt{7}$	✓ Determines magnitude of $\mathbf{c}$. ✓ Solves for x.	1.3.2

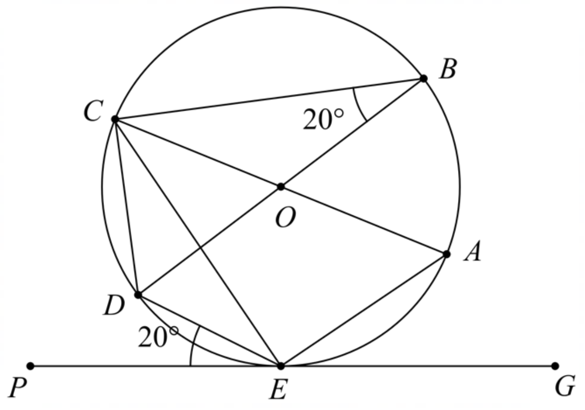
(f) Given $\mathbf{d} = 7\mathbf{i} + \mathbf{j}$, determine λ, such that $3\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{d}$. (3 marks)

Solution	Specific behaviours	Point
$(3\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{d} = 0$ $\left((-9 + 6\lambda)\mathbf{i} + (3 - 6\lambda)\mathbf{j}\right) \cdot (7\mathbf{i} + \mathbf{j}) = 0$ $36\lambda - 60 = 0$ $\lambda = \frac{5}{3}$	✓ Indicates dot product is equal to 0. ✓ Substitutes in correctly, and determines dot product. ✓ Determines λ.	1.3.11

Question 12 (10 marks)

In the diagram at right, A, B, C, D and E are five points on the circle with centre O.
AC and BD are diameters, and PEG is a tangent to the circle at E.

It is given that $\angle DEP = \angle DBC = 20°$.



(a) Determine, giving reasons, the size of the following angles:

(i) $\angle DEC$ (1 mark)

Solution	Specific behaviours	Point
$\angle DEC$ $= 20°$ (angle in the same arc)	✓ Determines angle with reasons.	1.1.8

(ii) $\angle DCE$ (1 mark)

Solution	Specific behaviours	Point
$\angle DCE$ $= 20°$ (angle alternate segment)	✓ Determines angle with reasons.	1.1.11

(b) Using part (a) and triangle CDE, explain why $\angle COD = \angle DOE$. (2 marks)

Solution	Specific behaviours	Point
$\overline{CD} = \overline{DE}$ (CDE is isoceles) $\therefore \angle COD = \angle DOE$ (equal chords subtend equal angles)	✓ Indicates ΔCDE is isosceles. ✓ Explains why $\angle COD = \angle DOE$.	1.1.10

(c) Prove that $\angle EAC = 40°$, giving reasons. (3 marks)

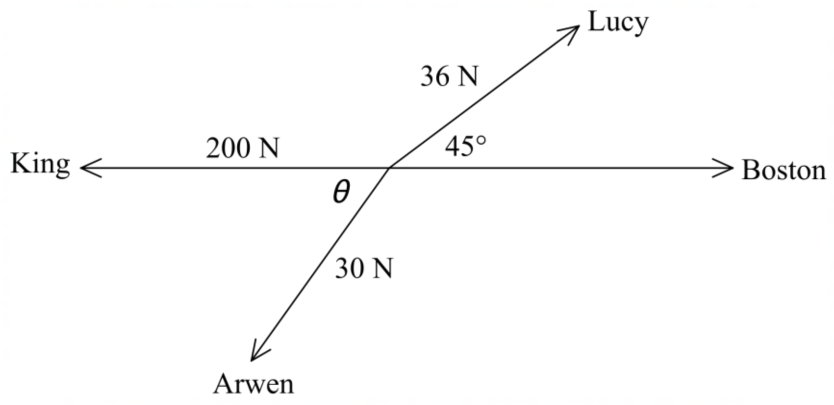
Solution	Specific behaviours	Point
$\angle AEC = 90°$ (angle in a semicircle) $\angle AEG = 50°$ (angle on straight line) $\angle ECA = 50°$ (angle alternate segment) $\angle EAC = 40°$ (angle sum in a triangle)	✓ Determines $\angle AEG = 50°$. ✓ Proves $\angle EAC = 40°$. ✓ States reasons.	1.1.6 1.1.11

(d) Is it possible to draw a circle through the points E, O, C and D. Justify your answer. (3 marks)

Solution	Specific behaviours	Point
$\angle OCD = 50° + 20° = 70°$ $\angle OEA = 90° - 50° = 40°$ $\angle OED = (90° - 40°) + 20° = 70°$ As $\angle OCD + \angle OED \neq 180°$ then $EOCD$ is not a cylic quadrilateral Hence, a circle cannot be drawn	✓ Determines $\angle OCD$. ✓ Determines $\angle OED$. ✓ Justifies that $EOCD$ is not cyclic.	1.1.14 1.1.15

Question 13 **(6 marks)**

Four dogs are arguing over a bone. The diagram shows the four dogs, the direction they are pulling in, and the force applied. The angle between King's force and Boston's force is $180°$.



If the bone does not move, determine

(a) the size of angle θ, (3 marks)

Solution	Specific behaviours	Point
Using **j** components: $36\sin 45° = 30\sin\theta$ $\theta = 58.05°$	✓ Recognises the need to look at **j** components. ✓ Determines force for Lucy and Arwen. ✓ Determines θ.	1.3.1 1.3.14

(b) the force that Boston pulls at. (3 marks)

Solution	Specific behaviours	Point
Comparing **i** components: $-200 - 30\cos 58.05 + b + 18\sqrt{2} = 0$ $b = 190.42$ N Boston $= \langle 190.42, 0\rangle$	✓ Compares **i** components. ✓ Determines b. ✓ States force.	1.3.1 1.3.14

Question 14 (11 marks)

(a) Two students wrote in their notebooks an expression the teacher had written on the board.

Student 1 wrote: $\forall x \in \mathbb{R}(\exists y \in \mathbb{R}, x < y)$

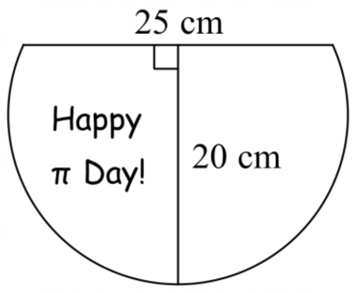
Student 2 wrote: $\exists x \in \mathbb{R}(\forall y \in \mathbb{R}, x < y)$

One student copied down the expression incorrectly. If the teacher said the statement was true, who wrote it down incorrectly? Justify your answer. (3 marks)

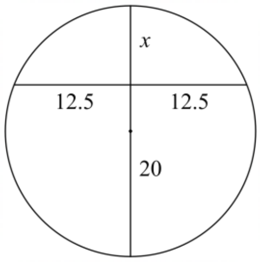
Solution	Specific behaviours	Point
Student 1′s statement is true as y could be $x+1$. Student 2′s statement is false as there is no number x which is less than all real numbers y. Hence student 2 made an error.	✓ Explains why first statement is true. ✓ Explains why second statement is false. ✓ Concludes Student 2 made a mistake.	1.1.4

(b) A teacher brought in cake for their class for Pi Day.

Unfortunately, someone had sliced off a segment of cake and eaten it.

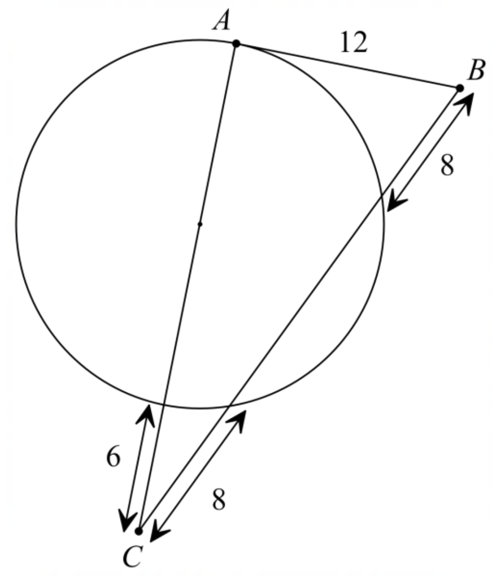


Use the given measurements to determine the diameter of the original cake, to one decimal place. (4 marks)

Solution	Specific behaviours	Point
 $20x = 12.5^2$ $x = 7.81$ Diameter $= 20 + 7.81 = 27.8$ cm	✓ Divides chord of 25 cm into two equal sections. ✓ Uses intersecting chord theorem. ✓ Determines missing length. ✓ Determines diameter.	1.1.12

(c) AB is a tangent to the circle given circle.

Determine the area of ΔABC, correct to one decimal place. (4 marks)



Solution (1)	Specific behaviours	Point
Let the length of the part of secant BC inside the circle be x $12^2 = (x+8)\times 8$ $x = 10$ $\text{diameter} = \sqrt{26^2 - 12^2}$ $\text{diameter} = 23.07$ $\text{Area} = \frac{1}{2} \times 23.07 \times 12 = 138.4 \text{ units}^2$	✓ Uses secant-tangent theorem. ✓ Determines length of secant in circle. ✓ Determines diameter. ✓ Determines area.	1.1.13
Solution (2)	**Specific behaviours**	**Point**
Let the length of the part of secant BC inside the circle be x $12^2 = (x+8)\times 8$ $x = 10$ $\cos B = \frac{12}{26}$ $B = 62.51°$ $\text{Area} = \frac{1}{2} \times 26 \times 12 \sin 62.51$ $= 138.4 \text{ units}^2$	✓ Uses secant-tangent theorem. ✓ Determines length of secant in circle. ✓ Determines angle. ✓ Determines area.	1.1.13

Question 15 **(11 marks)**

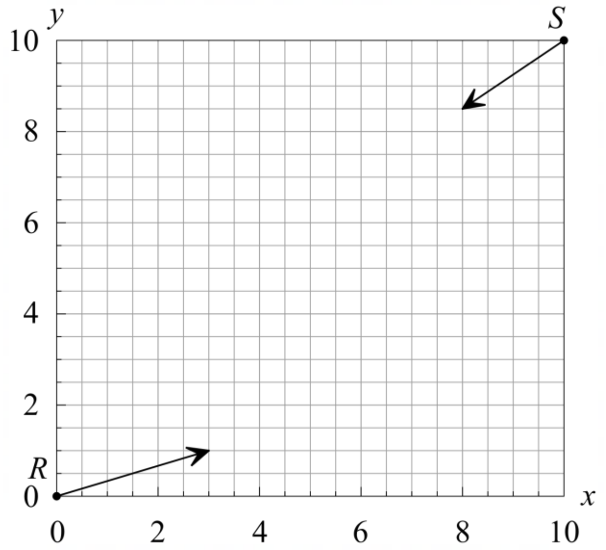
Two snails, Racer (R) and Speedy (S), are competing in a race.

The snails start in opposite corners of a square of side length 10 cm, where $(0, 0)$ represents the origin.

The initial position (in cm) and velocity vectors (in cms^{-1}) of the two snails are given by:

$$\mathbf{r}_R = \begin{pmatrix}0\\0\end{pmatrix} \quad \mathbf{v}_R = \begin{pmatrix}0.6\\0.2\end{pmatrix}$$
$$\mathbf{r}_S = \begin{pmatrix}10\\10\end{pmatrix} \quad \mathbf{v}_S = \begin{pmatrix}-0.4\\-0.3\end{pmatrix}$$

Let t represent the time in seconds since the snails left their respective corners.



The diagram above shows the starting positions, and the direction the snails are travelling in.

(a) Which snail is the fastest? Justify your answer using an appropriate calculation. (2 marks)

Solution	Specific behaviours	Point
$\lvert\mathbf{v}_R\rvert = 0.6325$ $\lvert\mathbf{v}_S\rvert = 0.5$ Racer is fastest	✓ Determines magnitude of velocity vectors. ✓ States Racer is fastest.	1.3.1 1.3.2

(b) Find the angle between the two snails' direction. (1 mark)

Solution	Specific behaviours	Point
$161.57°$	✓ Uses technology to determine angle.	1.3.10

(c) Determine the position of each snail after t seconds. (2 marks)

Solution	Specific behaviours	Point
$\mathbf{r}_R(t) = \begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}0.6\\0.2\end{pmatrix}$ $\mathbf{r}_S(t) = \begin{pmatrix}10\\10\end{pmatrix} + t\begin{pmatrix}-0.4\\-0.3\end{pmatrix}$	✓ Determines position vector for Racer. ✓ Determines position vector for Speedy.	1.3.14

The winner of the race is the first snail to reach the opposite corner.

Assuming the snails continue in the same direction until they hit the edge of the square, and then travel at the same speed towards the opposite corner,

(d) determine which snail wins the race. (6 marks)

Solution	Specific behaviours	Point
For Racer: Reaches edge when **i** component is 10 $10 = 0 + 0.6t$ $t = \frac{50}{3}$ secs $\mathbf{r}_R\left(\frac{50}{3}\right) = \begin{pmatrix} 10 \\ \frac{10}{3} \end{pmatrix}$ Distance to corner $= 10 - \frac{10}{3} = \frac{20}{3}$ m Time to corner $= \frac{20}{3} \div 0.6325 = 10.54$ Total time $= \frac{50}{3} + 10.54 = 27.21$ secs **For Speedy**: Reaches edge when **i** component is 0 $0 = 10 - 0.4t$ $t = 25$secs $\mathbf{r}_S(25) = \begin{pmatrix} 0 \\ 2.5 \end{pmatrix}$ Distance to corner $= 2.5$m Time to corner $= 2.5 \div 0.5 = 5$ Total time $= 25 + 5 = 30$ secs Racer wins the race	✓ Uses **i** component to determine time when each snail reaches edge. ✓ Determines position of each snail when it reaches the edge. ✓ Determines distance to corner. ✓ Uses speed from (a) to determine time to corner for each snail. ✓ Determines total time for each snail. ✓ Determines Racer wins the race.	1.3.14

Question 16 **(4 marks)**

A bag consists of 10 cards, numbered 0 to 9 inclusive.

(a) Show that if a student selects 7 of them, then there are two cards with a sum of 10.

(2 marks)

Solution	Specific behaviours	Point
Pairing the cards to make sums of 10 gives: $\begin{matrix} 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 6 \end{matrix}$ as well as 0 5 If the pairs, 0 and 5 are the pigeonholes, then as there are 6 pigeonholes, selecting 7 cards will give a sum of 10.	✓ Pairs cards to give a sum of 10. ✓ Uses pigeonhole principle to explain answer.	1.2.6

At least one card in the bag is now removed. The student does not know which card(s) have been removed.

(b) The student now selects **6 cards**. Determine the **minimum** number of cards that can be removed, to guarantee at least two of the six chosen cards sum to 10. Justify your answer.

(2 marks)

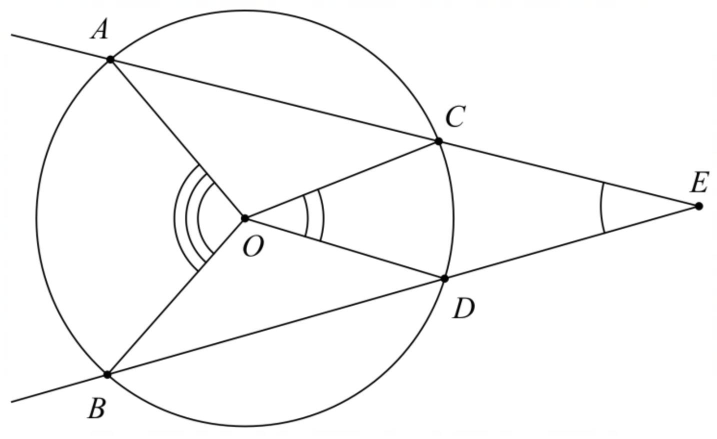
Solution	Specific behaviours	Point
As 6 cards are selected there must be 5 pigeonholes. These 5 pigeonholes could be: 3 pairs as well as 0 and 5 i.e. 8 cards 4 pairs as well as either 0 or 5 i.e. 9 cards As the student is unaware of the cards removed then two can be removed.	✓ States there needs to be 5 pigeonholes. ✓ Determines minimum number of cards that can be removed.	1.2.6

Question 17 **(11 marks)**

The secant angle theorem states that the angle formed by two intersecting secants is half of the difference of the two interior angles.

This is

$$\angle CED = \frac{\angle AOB - \angle COD}{2}$$



(a) (i) Let $\angle AOB = 2\theta$, then state the size of angle $\angle ACB$. (1 mark)

Solution	Specific behaviours	Point
$\angle ACB = \theta$	✓ Determines angle.	1.1.7

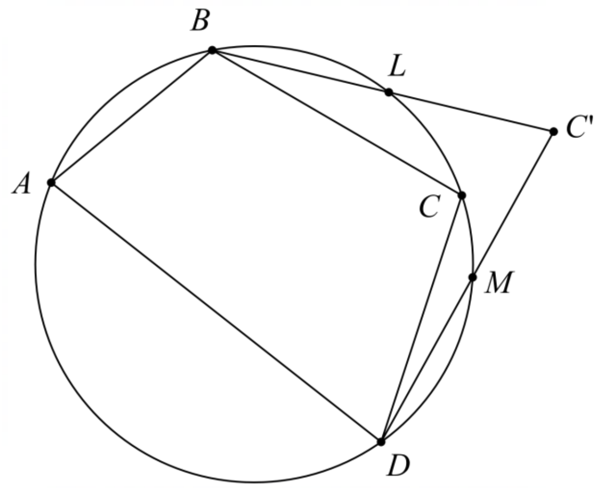
(ii) Let $\angle COD = 2\phi$, then state the size of angle $\angle CBD$. (1 mark)

Solution	Specific behaviours	Point
$\angle CBD = \phi$	✓ Determines angle.	1.1.7

(iii) Using ΔECB and parts (i) and (ii), prove the secant angle theorem. (3 marks)

Solution	Specific behaviours	Point
In triangle ΔECB $\angle ECB = 180° - \theta$ (straight line) $\angle CEB = 180° - (180° - \theta) - \phi$ (sum of angles in a triangle) $\angle CEB = \theta - \phi$ $\angle CEB = \frac{2\theta - 2\phi}{2}$ $\angle CEB = \frac{\angle AOB - \angle COD}{2}$	✓ Shows use of ΔECB. ✓ Determines expression in terms of θ and ϕ. ✓ Proves required result giving reasons.	1.1.15

In the diagram below $ABCD$ is a quadrilateral with $\angle ABC + \angle CDA = 180°$.



(b) Use the secant angle theorem from part (a), and proof by contradiction to prove that $ABCD$ is cyclic.

Hint: Prove that $\angle BCD$ must lie on the circumference. (6 marks)

<table>
<tr><th>Solution</th><th>Specific behaviours</th><th>Point</th></tr>
<tr><td>Assume C' lies outside the circle
Given $\angle ABC' + \angle C'DA = 180°$ (1)

By secant angle theorem:
$\angle BC'D = \dfrac{\text{reflex}\angle BOD - \angle LOM}{2}$
$\angle BC'D = 180 - \dfrac{\angle BOD}{2} - \dfrac{\angle LOM}{2}$ (2)

Centre angle is half angle at circumfrence:
$\angle BAD = \dfrac{\angle BOD}{2}$ (3)

$\angle BAD + \angle ABC' + \angle BC'D + \angle C'DA$
$= 360°$ (angles in a quadrilateral)

Substituting (1) and simplifying:
$\angle BAD + \angle BC'D = 180°$

Substituting (2) and (3):
$\dfrac{\angle BOD}{2} + 180 - \dfrac{\angle BOD}{2} - \dfrac{\angle LOM}{2} = 180°$
$\therefore \dfrac{\angle LOM}{2} = 0°$

If $\angle LOM = 0°$, then C' cannot lie outside the circle, which contradicts our assumption $\therefore$ $ABCD$ is cyclic</td><td>✓ States assumption.

✓ Shows use of the secant angle theorem.

✓ Expresses $\angle BAD$ in terms of $\angle BOD$.

✓ Uses (1) to derive relationship between $\angle BAD$ and $\angle BC'D$.

✓ Uses (2) and (3) to determine $\angle LOM = 0°$.

✓ Explains how $\angle LOM = 0°$ contradicts the assumption.</td><td>1.1.2
1.1.9</td></tr>
</table>